IL NUOVO CIMENTO Vol. ?, N. ? ?

General Aspects of the de Sitter phase

Giovanni Imponente ([1]) ([2]) ([3]) and Giovanni Montani ([4]) ([3])

([1]) *Dipartimento di Fisica, Universitá di Napoli "Federico II", Napoli – Italy*
([2]) *INFN – Sezione di Napoli*
([3]) *ICRA – International Center for Relativistic Astrophysics*
([4]) *Dipartimento di Fisica – G9 Universitá di Roma "La Sapienza", Roma – Italy*

Summary. —
We present a detailed discussion of the inflationary scenario in the context of inhomogeneous cosmologies. After a review of the fundamental features characterizing the inflationary model, as referred to a homogeneous and isotropic Universe, we develop a generalization in view of including small inhomogeneous corrections in the theory.
A second step in our discussion is devoted to show that the inflationary scenario provides a valuable dynamical "bridge" between a generic Kasner-like regime and a homogeneous and isotropic Universe in the horizon scale.
This result is achieved by solving the Hamilton-Jacobi equation for a Bianchi IX model in the presence of a cosmological space-dependent term.
In this respect, we construct a quasi-isotropic inflationary solution based on the expansion of the Einstein equations up to first-two orders of approximation, in which the isotropy of the Universe is due to the dominance of the scalar field kinetic term; the first order of approximation corresponds to the inhomogeneous corrections and is driven by the matter evolution.
We show how such a quasi-isotropic solution contains a certain freedom in fixing the space functions involved in the problem. The main physical issue of this analysis corresponds to outline the impossibility for the classical origin of density perturbations, due to the exponential decay of the matter term during the de Sitter phase.

PACS 98.80.Bp – Origin and formation of the Universe.
PACS 98.80.Cq – Inflationary Universe .

1. – Introduction

The homogeneity and isotropy of the Very Early Universe evolving towards the initial singularity shows instability for density perturbations [27].

We will discuss how to connect the Mixmaster dynamics and its properties with an inflationary scenario, as well as how such a scheme can be interpolated with a quasi-isotropic solution of the Einstein equations [14, 15, 16], in order to recover the homogeneous and isotropic Universe so far observed.

A significant degree of inhomogeneity is manifested at the scale of galaxies, clusters, etc., nevertheless on sufficiently large scales (of the order of 100 Mpc) inhomogeneities are smoothed out and isotropy and homogeneity are reached like in the Friedmann-Lemaitre-Robertson-Walker (FLRW) model. Even the early Universe exhibits a level of isotropy and homogeneity by far higher than now, as testified by the extreme uniformity (of the order of 10^{-4}) of the Cosmic Microwave Background Radiation (CMBR).

The homogeneity and isotropy of the Universe is well tested up to $10^{-3} - 10^{-2}$ seconds of its life by the very good agreement between the abundances of light elements as predicted by the Standard Cosmological Model (SCM) and the one actually observed [24]; furthermore, reliable indications support the idea that the Universe evolved through an inflationary scenario and since that age it reached isotropy and homogeneity on the horizon scale at least [9, 36].

Despite of these evidences favourable to the large scale homogeneity, many relevant features suggest that in the very early stage of evolution it had to be described by much more general inhomogeneous solutions of the Einstein equations. With respect to this we remark the following two issues:

(*i*) the backward instability of the density perturbations to an isotropic and homogeneous Universe [27] allows us to infer that, when approaching the initial singularity, the dynamics evolved as more general and complex models; thus the appearance of an oscillatory regime is expected in view of its general nature, *i.e.*because its perturbations correspond simply to redefine the spatial gradients involved in the Cauchy problem.

(*ii*) Since the Universe dynamics during the Planckian era underwent a quantum regime, then no symmetry restrictions can be *a priori* imposed on the cosmological model; in fact, the wave functional of the Universe can provide information on the casual scale at most, and therefore the requirement for a global symmetry to hold would imply a large scale correlation of different horizons. Thus, the quantum evolution of the Universe is appropriately described only in terms of a generic inhomogeneous cosmological model.

In this paper, we address two main topics concerning the Universe evolution: in first place, we show how the inflationary scenario provides the natural framework to link the inhomogeneous oscillatory regime with anisotropic and homogeneous Universe on the horizon scale; then, we investigate how a classical origin of inhomogeneous perturbations is ruled out out by a de Sitter phase in a quasi-isotropic scheme.

2. – The Standard Cosmological Model

The most general metric which is spatially homogeneous and isotropic is the FLRW one, which in terms of the *comoving coordinates* (t, r, θ, ϕ) reads

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{1}$$

where the scale factor $R(t)$ is a generic function of time only and, for an appropriate rescaling of the coordinates, the factor $k = 0, \pm 1$ distinguishes the sign of constant spatial curvature.
Such coordinates represent a reference frame participating in the expansion of the Universe: an observer at rest will remain at rest, leaving the effects of the expansion to the cosmic scale factor $R(t)$.

The FLRW dynamics is reduced to the time dependence of the scale factor $R(t)$ once solved the Einstein equations with a stress-energy tensor $T_{\mu\nu}$ for all the fields present (matter, radiation, etc.) which then must be diagonal, with the spatial components equal to each other for the homogeneity and isotropy constraints. A simple realization is given by the perfect fluid one, characterized by a space-independent energy density $\rho(t)$ and pressure $p(t)$

$$T^{\mu}{}_{\nu} = \mathrm{diag}(\rho, -p, -p, -p)\,, \tag{2}$$

and in this case the $0-0$ component of the Einstein equations reads

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3}\rho \tag{3}$$

which is the *Friedmann equation*, while the $i-i$ components are

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G p\,. \tag{4}$$

The difference between (3) and (4) leads to

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)\,, \tag{5}$$

which is solved for $R(t)$ once provided an equation of state, *i.e.*a relation between ρ and p.
When the Universe was radiation-dominated, as in the early period, the radiation component provided the greatest contribution to its energy density and for a photon gas we have $p = \rho/3$.
The present-time Universe is, on the contrary, matter-dominated: the "particles" (*i.e.*the galaxies) have only long-range (gravitational) interactions and can be treated as a pressureless gas ("dust"): the equation of state is $p = 0$.

Nevertheless, the energy tensor appearing in the Einstein field equations describes the complete local energy due to all non-gravitational fields, while gravitational energy has a non-local contribution. An unambiguous formulation for such a non-local expression is found only in the expressions used at infinity for an asymptotically flat space-time [34, 2]. This is due to the property of the mass-energy term to be only one component of the energy-momentum tensor which can be reduced only in a peculiar case to a four-vector expression which can not be summed in a natural way.

Bearing in mind such difficulties, the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ leads to

$$d(\rho R^3) = -p d(R^3) \tag{6}$$

*i.e.*the first law of thermodynamics in an expanding Universe which, via the equation of state, leads to a differential relation for $\rho(R)$. If we take its solution together with the Friedmann equation (3) corresponding to $k = 0$, then we find the following behaviours

$$\text{RADIATION} \qquad \rho \propto R^{-4}\,, \quad R \propto t^{1/2} \tag{7a}$$

$$\text{MATTER} \qquad \rho \propto R^{-3}\,, \quad R \propto t^{2/3} \tag{7b}$$

where as long as the Universe is not curvature-dominated, *i.e.*for sufficiently small values of R, the choice of $k = 0$ is not relevant. The equation of state $p = -\rho$ leads to $\rho =$ const. and $R \propto e^{At}$ $A = const.$, *i.e.*a phase of exponential expansion, equivalent to adding a constant term to the right-hand side of the Einstein equation mimicking a cosmological constant: this is exactly what the inflationary paradigm proposes to overcome the paradoxes of the standard model outlined in the next Section **3**.
The Hubble parameter $H \equiv \dot{R}/R$ and the critical density $\rho_c \equiv 3H^2/8\pi G$ make it possible to rewrite (3) as

$$\frac{k}{H^2R^2} = \frac{\rho}{3H^2/8\pi G} - 1 \equiv \Omega - 1, \tag{8}$$

where Ω is the ratio of the density to the critical one $\Omega \equiv \rho/\rho_c$; since H^2R^2 is always positive, the relation between the sign of k and the sign of $(\Omega - 1)$ reads

$$k = +1 \quad \Rightarrow \quad \Omega > 1 \qquad \text{CLOSED} \tag{9a}$$

$$k = 0 \quad \Rightarrow \quad \Omega = 1 \qquad \text{FLAT} \tag{9b}$$

$$k = -1 \quad \Rightarrow \quad \Omega < 1 \qquad \text{OPEN} \tag{9c}$$

3. – Shortcomings of the Standard Model: Horizon and Flatness Paradoxes

Despite the simplicity of the Friedmann solution (in view also of the thermodynamic property which can be studied in detail) some paradoxes occur when taking into account the problem of *initial conditions*. The observed Universe has to match very specific physical conditions in the very early epoch, but [8] showed that the set of initial data that can evolve to a Universe similar to the present one is of zero measure and the standard model tells *nothing* about initial conditions.

. – Flatness

Let us assume that all particle species present in the early Universe have the same temperature as the photon bath, *i.e.*$T_i = T_\gamma$ and are far from any mass threshold. Then the average number of degrees of freedom for the photons and fermions bath g^* is a constant and $T \propto R^{-1}$ [24]. The average energy density

$$\rho = \frac{\pi^2}{30} g^* T^4 \tag{10}$$

substituted in the (3) reads

$$\left(\frac{\dot{T}}{T}\right)^2 + \epsilon(T)T^2 = \frac{4\pi^3}{45} G g^* T^4\,, \tag{11}$$

where

$$\epsilon(T) \equiv \frac{k}{R^2T^2} = k\left[\frac{2\pi^2}{45}\frac{g^*}{S}\right]^{2/3}, \tag{12}$$

and since $S = R^3 s$ is the total entropy per comoving volume the entropy density reads

$$s = \frac{2\pi^2}{45} g^* T^3 . \tag{13}$$

Today $\rho \simeq \rho_c$, then by taking $\rho < 10\rho_c$ in (8) we have

$$\left|\frac{k}{R^2}\right| < 9H^2 . \tag{14}$$

For $k = \pm 1$ (the case $k = 0$ is regained in the limit $R \to \infty$), we have for today $R > \frac{1}{3}H^{-1} \approx 3 \cdot 10^9$ years and $T_\gamma \simeq 2.7$ K; from (12), the present photon contribution to the entropy has the lower bound

$$S_\gamma > 3 \cdot 10^{85} \tag{15}$$

expressed in units of the Boltzmann constant $k_B = 1.3806 \cdot 10^{-16}$ erg/K. The relativistic particles present today together with photons are the three neutrino species and their contribution to the total entropy is of the same order of magnitude

$$S > 10^{86} , \tag{16}$$

and finally with

$$|\epsilon| < 10^{-58} g^{*2/3} \tag{17}$$

we gain

$$\left|\frac{\rho - \rho_c}{\rho}\right| = \frac{45}{4\pi^3}\frac{m_P^2}{g^* T^2}|\epsilon| < 3 \cdot 10^{59} g^{*-1/3}\left(\frac{m_P}{T}\right)^2 , \tag{18}$$

where m_P is the Planck mass $(\hbar c/G)^{1/2} = 2.1768 \cdot 10^{-5}$g $= 1.2211 \cdot\ 10^{19}$ GeV.
When taking $T = 10^{17}$ GeV, all species in the standard model of particle interactions – 8 gluons, $W^\pm$, Z^0, 3 generations of quark and leptons – are relevant and relativistic: then $g^* \approx 100$ and finally

$$\left|\frac{\rho - \rho_c}{\rho}\right|_{T=10^{17}\mathrm{GeV}} < 10^{-55} . \tag{19}$$

A flat Universe today requires Ω of the original one close to unity up to a part in 10^{55}. A little displacement from flatness at the beginning – for example 10^{-30} – would produce an actual Universe either very open or very closed, so that $\Omega = 1$ is a very unstable condition: this is the *flatness problem*.

The natural time scale for cosmology is the Planck time ($\sim 10^{-43}$ sec): in a time of this order a typical closed Universe would reach the maximum size while an open one would become curvature dominated. The actual Universe has survived 10^{60} Planck times without neither recollapsing nor becoming curvature dominated.

. – Horizon

Neglecting the ϵT^2 term, (11) is solved by

$$T^2 = \left(\frac{4\pi^3}{45} g^* \right)^{-1/2} \frac{m_P}{2\,t} \,. \tag{20}$$

A light signal emitted at $t = 0$ travelled during a time t the physical distance

$$l(t) = R(t) \int_0^t \frac{dt'}{R(t')} = 2\,t \tag{21}$$

in a radiation-dominated Universe with $R \propto t^{1/2}$, measuring the physical horizon size, *i.e.*the linear size of the greatest region causally connected at time t. The distance (21) has to be compared with the radius $L(t)$ of the region which will evolve in our currently observed region of the Universe. Conservation of entropy for $s \propto T^3$ gives

$$L(t) = \left(\frac{s_0}{s(t)} \right)^{1/3} L_0 \,, \tag{22}$$

where s_0 is the present entropy density and $L_0 \sim H^{-1} \simeq 10^{10}$ years is the radius of the currently observed region of the Universe. The ratio of the volumes provides

$$\frac{l^3}{L^3} = 4 \cdot 10^{-89} g *^{-1/2} \left(\frac{m_P}{T} \right)^3 \tag{23}$$

and, as above, for $g^* \sim 100$ and $T \sim 10^{17}$GeV we obtain

$$\left. \frac{l^3}{L^3} \right|_{T=10^{17}\mathrm{GeV}} \sim 10^{-83} \,. \tag{24}$$

The currently observable Universe is composed of several regions which have *not* been in causal contact for the most part of their evolution, preventing an explanation about the present days Universe smoothness. In particular, the spectrum of the CMBR is uniform up to 10^{-4}. Moreover, we have at the time of recombination, *i.e.*when the photons of the CMBR last scattered, the ratio $l^3/L^3 \sim 10^5$: the present Hubble volume consists of about 10^5 causally disconnected regions at recombination and no process could have smoothed out the temperature differences between these regions without violating causality. The particle horizon at recombination subtends an angle of only 0.8° in the sky today, while the CMBR is uniform across the sky.

4. – The Inflationary Paradigm

The basic ideas of the theory of inflation rely firstly on the original work by [11], *i.e.*the *old inflation*, which provides a phase in the Universe evolution of exponential expansion; then formulation of *new inflation* by [28] introduced the slow-rolling phase in inflationary dynamics; finally, many models have sprung from the original theory.

In [11] is described a scenario capable of avoiding the horizon and flatness problems: both paradoxes would disappear dropping the assumption of adiabaticity and in such a case the entropy per comoving volume s would be related as

$$s_0 = Z^3 s_{\rm early} \tag{25}$$

where s_0 and $s_{\rm early}$ refer to the values at present and at very early times, for example at $T = T_0 = 10^{17}$ GeV, and Z is some large factor.
With this in mind, the right-hand side of (18) is multiplied by a factor Z^2 and the value of $|\rho - \rho_c|/\rho$ would be of the order of unity if

$$Z > 3 \cdot 10^{27}\,, \tag{26}$$

getting rid of the flatness problem.
The right-hand side of (22) is multiplied by Z^{-1}: for any given temperature, the length scale of the early Universe is smaller by a factor Z than previously evaluated, and for Z sufficiently large the initial region which has evolved in our observed one would have been smaller than the horizon size at that time.
Let us evaluate Z considering that the right-hand side of (22) is multiplied by Z^3: if

$$Z > 5 \cdot 10^{27}\,, \tag{27}$$

then the horizon problem disappears.
Making some *ad hoc* assumptions, the model accounts for the horizon and flatness paradoxes while a suitable theory needs a physical process capable of such a large entropy production. A simple solution relies on the assumption that at very early times the energy density of the Universe was dominated by a scalar field $\phi(\vec{x}, t)$, *i.e.*$\rho = \rho_\phi + \rho_{\rm rad} + \rho_{\rm mat} + \dots$ with $\rho_\phi \gg \rho_{\rm rad,\ mat,\ etc}$ and hence $\rho \simeq \rho_\phi$.

The quantum field theory Lagrangian density for such a field is

$$\mathcal{L} = \frac{\partial^\mu\phi\,\partial_\mu\phi}{2} - V(\phi) \tag{28}$$

and the corresponding stress-energy tensor

$$T^\mu_{\ \nu} = \partial^\mu\phi\,\partial_\nu\phi - \mathcal{L}\delta^\mu_{\ \nu} \tag{29}$$

by which for a spatially homogeneous and isotropic Universe the form of a perfect fluid leads to [26]

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) + \frac{1}{2R^2}\nabla^2\phi\,, \tag{30a}$$

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) - \frac{1}{6R^2}\nabla^2\phi\,. \tag{30b}$$

Spatial homogeneity would induce a slow variation of ϕ with position, hence the spatial gradients are negligible and the ratio $\omega = p/\rho$ reads

$$\frac{p_\phi}{\rho_\phi} \simeq \frac{\frac{\dot{\phi}^2}{2} + V(\phi)}{\frac{\dot{\phi}^2}{2} - V(\phi)} . \tag{31}$$

If the field is at a minimum of the potential, $\dot{\phi} = 0$, and (31) becomes an equation of state

$$p_\phi = -\rho_\phi \tag{32}$$

giving rise to a phase of exponential growth of $R \propto e^{Ht}$, the *inflationary* or *de Sitter phase.*

The field evolution is very different when in vacuum or in a thermal bath and such a coupling can be summarized by adding a term $-(1/2)\lambda T^2\phi^2$ to the Lagrangian. The potential $V(\phi)$ is replaced by the *finite-temperature* effective potential

$$V_T(\phi) = V(\phi) + \frac{1}{2}\lambda T^2\phi^2 . \tag{33}$$

In the old inflation, $V(\phi)$ appearing in (33) has the form of a Georgi-Glashow $SU(5)$ theory

$$V(\phi) = \frac{1}{4}\phi^4 - \frac{1}{3}(A+B)\phi^3 + \frac{1}{2}AB\phi^2 \tag{34}$$

with $A > 2B > 0$, and possesses a local minimum at $\phi = 0$ and a global minimum at $\phi = A$, separated by a barrier with a maximum at $\phi = B$. The temperature-dependent term $-(1/2)\lambda T^2\phi^2$ leaves the local minimum unchanged and raises the global minimum as well as the maximum – the former by a larger amount than the latter.

At sufficiently high temperature $V_T(\phi)$ has only one global minimum at $\phi = 0$ and, as long as T decreases, a second minimum develops at $\phi = \sigma(T)$, with $V(0) < V(\sigma)$, and $\phi = 0$ is still the true minimum of the potential.

At a certain critical temperature T_c the two minima are exactly degenerate as $V(0) = V(\sigma)$ and at temperatures below T_c, $V(\sigma) < V(0)$ and $\phi = 0$ is no longer the true minimum of the potential.

Let us consider when at some initial time, corresponding to $T = T_i > T_c$, the field ϕ is trapped in the minimum at $\phi = 0$ (*false vacuum*) with constant energy density, given by (30a) $V_T(0) \simeq T_C^4$. The temperature lowers with the Universe expansion up to the critical value T_c: the scalar field begins dominating the Universe and a second minimum of the potential develops at $\phi = \sigma$. The inflationary phase is characterized by

$$R(t) \propto e^{Ht} \tag{35}$$

where the Hubble parameter $H \equiv \dot{R}/R$ is a constant; $\phi = 0$ becomes a metastable state, since there exists a more energetically favourable one at $\phi = \sigma$ (*true vacuum*).

The potential barrier lying within cannot prevent a non-vanishing probability per unit time that the field performs a first order phase transition via quantum tunnelling to the true vacuum state, proceeding along by bubble nucleation: bubbles of the true vacuum phase are created expanding outward at the speed of light in the surrounding "sea" of false vacuum, until all the Universe has undergone the phase transition.
If the rate of bubble nucleation is low, the time to complete the phase transition can be very long if compared to the expansion time scale: when the transition ends, the Universe has cooled to a temperature T_f many orders of magnitude lower than T_c.
On the contrary, when approaching the true vacuum, the field ϕ begins to oscillate around this position on a time scale short if compared to the expansion one, releasing all its vacuum energy in the form of ϕ-particles, the quanta of the ϕ field. The oscillations are damped by particles decay and, when the corresponding products thermalize, the Universe is reheated to a temperature T_r of the order of T_c. This represents the release of the latent heat associated with the phase transition after which the scalar field is no longer the dominant component of the Universe: inflation comes to an end and the standard FLRW cosmology is recovered.

The non-adiabatic reheating process releases an enormous amount of entropy whose density is increased by a factor of the order of $(T_r/T_f)^3 \simeq (T_c/T_f)^3$, while R remains nearly constant and the entropy is increased by a factor $(T_c/T_f)^3$ too.
Flatness and horizon paradoxes are solved if the Universe super-cools of 28 or more orders of magnitude during inflation. Even if this looks very difficult to achieve, it is enough that the transition takes place in about a hundred Hubble times: the inflationary expansion is adiabatic and the entropy density $s \sim R^{-3}$. Since $s \propto T^3$, then $T \propto R^{-1} \propto e^{-Ht}$ and finally

$$\frac{T_c}{T_f} = e^{H\Delta t}\,, \tag{36}$$

where Δt is the duration of the de Sitter phase. From the requirement $(T_c/T_f) > 10^{28}$ it follows

$$\Delta t > 50H^{-1}\,. \tag{37}$$

The critical temperature is estimated to be of order of the energy typically involved for Grand Unification Theory spontaneous symmetry-breaking phase transition, 10^{14} GeV, and

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}V(0) \simeq \frac{T_c^4}{m_P^2} \qquad \rightarrow \qquad H^{-1} \simeq 10^{-34}\mathrm{s}\,. \tag{38}$$

The inflation removes the discussed paradoxes if the transition takes place in a time $t \approx 10^{-32}$s, nevertheless leaving some open problems regarding its dynamics:

- *a* - in the old scenario inflation never ends, due to smallness of the tunnelling transition rate, so that the nucleation of true vacuum bubbles is rare;

- *b* - the energy released during the reheating is stored in the bubbles kinetic energy so that the reheating proceeds via bubble collisions which remain too rare due to

low transition rate to produce sufficient reheating: the phase transition is never completed;

- *c* - such a discontinuous process of bubble nucleation via quantum tunnelling should produce a lot of inhomogeneities which aren't actually observed.

4˙1. *New Inflation: the Slow Rolling Model.* – In 1982, both [28] and [1] proposed a variant of Guth's model, now referred to as *new inflation* or *slow-rolling inflation*, in order to avoid the shortcomings of the old inflation. Their original idea considered a different mechanism of symmetry breaking, the so-called Coleman-Weinberg (CW) one. The potential of the CW model for a gauge boson field with a vanishing mass reads as

$$V(\phi) = \frac{B\sigma^4}{2} + B\phi^4\left[\ln\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right], \tag{39}$$

where B is connected to the fundamental constants of the theory and is $\simeq 10^{-3}$, while σ gives the energy associated with the symmetry breaking process and is $\simeq 2 \cdot 10^{15}$ GeV. The finite-temperature effective potential is obtained as above in (33) by adding a term of the form $(1/2)\lambda T^2\phi^2$. Expression (39) can be generalized by adding a mass term of the form $-(1/2)m^2\phi^2$. Defining a temperature-dependent "mass"

$$m_T \equiv \sqrt{-m^2 + \lambda T^2}\,, \tag{40}$$

the temperature-dependent potential becomes

$$V_T(\phi) = \frac{B\sigma^4}{2} + B\phi^4\left[\ln\left(\frac{\phi^2}{\sigma^2}\right) - \frac{1}{2}\right] + \frac{1}{2}m_T^2\phi^2\,. \tag{41}$$

The quantity ${m_T}^2$ can be used to parametrize the potential (41):

1. when ${m_T}^2 > 0$, the point $\phi = 0$ is a minimum of the potential, while when $m_T^2 < 0$ it is a maximum;
2. when ${m_T}^2 < \dfrac{4\sigma^2}{e} \simeq 1.5\sigma^2$, a second minimum develops for some $\bar{\phi} > 0$; initially this minimum is higher than the one at 0, but when m_T becomes lower than a certain value m_T^* ($0 < m_T^* < 1.5\sigma^2$) it will eventually become the global minimum of the potential.

If at some initial time the ϕ-field is trapped in the minimum at $\phi = 0$ the true minimum of the potential can eventually disappear as the temperature lowers. In this case, as m_T approaches 0, the potential barrier becomes low and can be easily overcome by *thermal* (not quantum) tunnelling, *i.e.*due to classical (thermal) fluctuations of the ϕ field around its minimum; the barrier can disappear completely when $m_T = 0$. Independently of what really happens, the phase transition doesn't proceed via a quantum tunnelling – a very discontinuous and a strongly first order process – but it evolves either by a *weakly* first order (thermal tunnelling) or second order (barrier disappearing at $m_T = 0$).
Hence the transition occurs rather smoothly, avoiding the formation of undesired inhomogeneities: inflation is not yet started, so the requirement for the field to take a long time to escape the false vacuum is not necessary; the transition rate can be very close to

unity, and completed without problem.
When the ϕ-field has passed the barrier (if any), it begins to evolve towards its true minimum. However, the potential (41) has a very interesting feature: if the coefficient of the logarithmic term is sufficiently high, the potential is very flat around 0, and then the field ϕ "slow rolls" rather than falling abruptly in the true vacuum state: during this slow roll phase the inflation takes place, lasting enough to produce the required super-cooling, as seen in the previous Section. When the field reaches the minimum, it begins to oscillate around it thus originating the reheating.
The problems of Guth's originary model are skipped moving the inflationary phase *after* the field has escaped the false vacuum state, by adding the slow-rolling phase.
Virtually all models of inflation are based upon this principle.

5. – The Bridge Solution

An inflationary scenario is of crucial importance to understand how an anisotropic Universe like the one described by the Bianchi type IX cosmological solution can approach an isotropic Universe when the volume expands enough.
In fact, during the inflation, the dominant term in the Einstein equations corresponds to the effective cosmological constant associated with the false-vacuum energy; such a term is an isotropic one and, when it dominates, it produces an exponential decay of the Universe anisotropies.
In this Section we show how it is possible to interpolate a Kasner-like behaviour with an isotropic stage of evolution. We will refer to this scheme as the *bridge solution* as in [23], because it allows to match the chaotic dynamics of the Bianchi IX model together with the later isotropic dynamics of the SCM.

With respect to this, let us observe that in the presence of an effective cosmological constant the action of the Bianchi IX framework takes the form

$$\delta I = \delta \int \left(p_\alpha \alpha' + p_+ {\beta_+}' + p_- {\beta_-}' + p_\phi \phi' - \sqrt{\frac{3\pi}{2}} N e^{-3\alpha} \mathcal{H} \right) d\eta = 0 \tag{42}$$

where we recall that $\mathcal{H}$ reads as

$$2\mathcal{H} = -p_\alpha^2 + p_+^2 + p_-^2 + p_\phi^2 + \mathcal{V} + e^{6\alpha}\Lambda\,, \tag{43}$$

Λ denotes a constant term and the Bianchi IX potential is [30]

$$\mathcal{V} = \frac{e^{4\alpha}}{3}\left\{ e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+}\left[\cosh(4\sqrt{3}\beta_-) - 1\right]\right\}. \tag{44}$$

The variation of the action (42) with respect to N provides the super-hamiltonian constraint to be $\mathcal{H} = 0$.
Near the Big Bang, $\alpha \to -\infty$, the Bianchi IX potential (44) turns out to be negligible with respect to the cosmological constant term and then, by replacing the conjugate momenta as

$$p_X \to \frac{\partial I}{\partial X}\,, \qquad X = \alpha, \beta_\pm, \phi \tag{45}$$

we get the Hamilton-Jacobi equation

$$-\left(\frac{\partial I}{\partial \alpha}\right)^2 + \left(\frac{\partial I}{\partial \beta_+}\right)^2 + \left(\frac{\partial I}{\partial \beta_-}\right)^2 + \left(\frac{\partial I}{\partial \phi}\right)^2 + e^{3\alpha}\Lambda = 0\,. \tag{46}$$

The general solution of (46) takes the form

$$I(\chi^r, \alpha) = \sum_r K_r \chi^r + \frac{2}{3}K_\alpha + \frac{K}{3}\ln\left|\frac{K_\alpha - K}{K_\alpha + K}\right|\,, \tag{47}$$

where $\chi_r = \{\beta_+, \beta_-, \phi\}$, K_r are constants of integration and

$$K = \sqrt{\sum_r K_r^2} \tag{48}$$

in which the index r is the label for $\beta_\pm, \phi$, while

$$K_\alpha(K_r, \alpha) = \pm\sqrt{\sum_r K_r^2 + 6\Lambda \exp{(3\alpha)}}\,, \tag{49}$$

by which we adopt the negative sign in order to describe Universe expansion; in fact, we have

$$\frac{\partial \alpha}{\partial t} = -\sqrt{\frac{3\pi}{2}} N e^{-3\alpha} p_\alpha\,. \tag{50}$$

According to the Hamilton-Jacobi method, we differentiate the action with respect to the quantities K^r and then, by putting the resulting expressions equal to arbitrary constant functions as

$$\frac{\delta I}{\delta K^r} = \chi_0^r = \text{const.}\,, \tag{51}$$

we find the solutions describing the trajectories of the system to be

$$\chi^r(\alpha) = \chi_0^r + \frac{K_r}{3\,|K|}\ln\left|\frac{K_\alpha - K}{K_\alpha + K}\right|\,. \tag{52}$$

Let us now consider the two opposite limits:

$\alpha \to -\infty$ we find the solution

$$\chi^r(\alpha) = \chi_0^r - \frac{K_r}{K}(\alpha - \alpha_0) \tag{53}$$

corresponding to a Kasner-like behaviour which can be regarded as the *last epoch* of the oscillatory regime [3, 12];

$\alpha \to +\infty$ we get the isotropic stage of evolution

$$\chi^r(\alpha) \to \chi^r_0 \,. \tag{54}$$

In fact, when the anisotropies $\beta_\pm$ approach constant values, they are no longer dynamical degrees of freedom and the solution looks homogeneous and isotropic. In the same limit, the scalar field frozens to a constant value too and it disappears from the dynamics as soon as the inflation starts.

Our analysis provides an interpolation between the two relevant stages of the Universe evolution and is a convincing feature in favour of the idea that inflation can be a mechanism to gain isotropy. In this sense, the inflationary scenario constitutes the natural mechanism by which the chaotic dynamics of the Bianchi IX model can be smoothed out towards a closed FLRW dynamics.

As shown in [23] the results of this Section can be extended to the generic inhomogeneous solution, *i.e.*point by point in space the same behaviour above outlined takes place.

From a cosmological point of view this means that after inflation the Universe reaches an high degree of homogeneity and isotropy within each horizon size.

6. – Quasi-isotropic Cosmological Solution

The perturbations to matter distribution not affecting uniformity are damped with time or remain constant in the isotropic model [27]. Hence, the evolution backwards in time of small density perturbations is of particular interest when considering cosmological models more general then the homogeneous and isotropic one, the assumption of uniformity being justified only at an approximate level.

The Friedmann solution is a particular case belonging to the class in which space contracts in a quasi-isotropic way, in the sense that linear distances change with the same time-dependence in all directions and such a solution existing only in a space filled with matter.

When considered the isotropic solution in the synchronous reference frame, isotropy and homogeneity are reflected in the vanishing of the off-diagonal metric components $g_{0\alpha}$. The approach to zero of such functions depends upon the matter equation of state: for the ultra-relativistic equation $p = \epsilon/3$, the metric is linear in t, hence $g_{\alpha\beta}$ is supposed to be expandable in integer powers of t.
Corresponding to these assumptions, the Einstein equations reduce to the partial differential system

$$\frac{1}{2}\partial_t k^\alpha_\alpha + \frac{1}{4}k^\beta_\alpha k^\alpha_\beta = \frac{\epsilon}{3}\left(4u_0u^0 + 1\right) \tag{55a}$$

$$\frac{1}{2}\left(k^\beta_{\alpha;\beta} - k^\beta_{\beta;\alpha}\right) = \frac{4}{3}\epsilon u_\alpha u^0 \tag{55b}$$

$$\frac{1}{2\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma}k^\beta_\alpha\right) + P^\beta_\alpha = \frac{\epsilon}{3}\left(u_\alpha u^\beta + \delta^\beta_\alpha\right) \,, \tag{55c}$$

where $P^\beta_\alpha = \gamma^{\beta\gamma}P_{\alpha\gamma}$ represents the three-dimensional Ricci tensor obtained by the metric $\gamma_{\alpha\beta}$ and u_i $(i = 0, 1, 2, 3)$ denotes the matter four-velocity vector field.

Let us consider a spatial metric of the form

$$g_{\alpha\beta} = t\ a_{\alpha\beta} + t^2\ b_{\alpha\beta} + \dots\ , \tag{56}$$

whose inverse reads as

$$g^{\alpha\beta} = t^{-1} a^{\alpha\beta} - b^{\alpha\beta} + \dots\ , \tag{57}$$

being $a^{\alpha\beta}$ the inverse tensor to $a_{\alpha\beta}$ which is the one used for the operations of rising and lowering indices as well as for the covariant differentiation. Once (56) is substituted in the field equations (55), we find the energy density and four-velocity to leading order as

$$\epsilon = \frac{3}{4t^2} - \frac{b}{2t} \tag{58a}$$

$$u_\alpha = \frac{t^2}{2}\left(b_{;\alpha} - b^{\beta}_{\ \alpha;\beta}\right)\ , \tag{58b}$$

respectively. The three-dimensional Christoffel symbols and the tensor $P_{\alpha\beta}$ are, to first approximation, independent of time and (55c) gives

$$P_\alpha^{\ \beta} + \frac{3}{4} b_\alpha^{\ \beta} + \frac{5}{12}\delta_\alpha^{\ \beta} b = 0 \tag{59}$$

and then

$$b_\alpha^{\ \beta} = -\frac{4}{3} P_\alpha^{\ \beta} + \frac{5}{18}\delta_\alpha^{\ \beta} P\,. \tag{60}$$

The six functions $a_{\alpha\beta}$ are arbitrary and, once they are given, the coefficients $b_{\alpha\beta}$ are determined by (60) and hence also the density of matter and its velocity can be derived. When $t \to 0$ the distribution of matter becomes homogeneous and its density approaches a value which is coordinate independent. The expression giving the distribution of velocity follows from (58b) explicitly as

$$u_\alpha = \frac{t^2}{9} b_{;\alpha}\,. \tag{61}$$

Such a framework is completed considering that an arbitrary transformation of the spatial coordinates (for example to reduce $a_{\alpha\beta}$ to a diagonal form) leaves only three arbitrary functions allowed in this quasi-isotropic solution, while the fully isotropic model is recovered in the specific choice of $a_{\alpha\beta}$ corresponding to the space of constant curvature $P_{\alpha\beta} = \text{const.} \times \delta_{\alpha\beta}$.

7. – Quasi-isotropic Solution Towards Singularity with a Scalar Field

In this Section we show how, following [31, 32], the quasi-isotropic Universe dynamics in presence of ultra-relativistic matter and a real self-interacting scalar field behaves in the asymptotic limit to the cosmological singularity, while in the next Section, following [14] we will see the opposite limit far from the singularity, opening the way also to a generalization [15, 17].

In particular, the presence of the scalar field kinetic term allows the existence of a quasi-isotropic solution characterized by an arbitrary spatial dependence of the energy density associated with the ultra-relativistic matter. To leading order, there is no direct relation between the isotropy of the Universe and the homogeneity of the ultra-relativistic matter in it distributed.

However, as shown in [3, 5] (see also [21]), the general behaviour of the Universe near the initial Big-Bang is characterized by a completely disordered dynamics and an increasing degree of anisotropy, up to develop a fully turbulent regime.

Hence, the contrast between such a general tendency to anisotropy and the evidence that in the forward evolution since a given age the Universe should have performed an highly symmetric behaviour is a problem related to properties of the Universe evolution at very different stages of anisotropy.

The quasi-isotropic solution allows, far enough from the initial singularity, the oscillatory regime [3, 5] to be decomposed in terms of a quasi-isotropic component plus suitable wave-like small corrections. An analogous decomposition has been obtained in [10] for the Bianchi type IX model as a homogeneous prototype of the general inhomogeneous case.

Here we summarize the features acquired by a quasi-isotropic solution (*i.e.*one in which the three spatial directions are dynamically equivalent) in presence of ultra-relativistic matter and a real self-interacting scalar field. Then a quasi-isotropic model solution exists and is characterized, asymptotically to the Big Bang, by an arbitrary distribution of the ultra-relativistic matter and in which the spatial curvature component has no dynamical role to first two orders of approximation. The presence of the scalar field kinetic term, close enough to the singularity, modifies deeply the general cosmological solution leading to the appearance of a dynamical regime characterized, point by point in space, by the monotonical collapse of the three spatial directions [4, 22].

Let us consider a synchronous reference frame in which the line element reads as

$$ds^2 = c^2 dt^2 - \gamma_{\alpha\beta}(t, x^\gamma) dx^\alpha dx^\beta \,, \tag{62}$$

the matter is described by a perfect fluid with ultra-relativistic equation of state $p = \frac{\epsilon}{3}$ and the scalar field $\phi(t, x^\gamma)$ admits a potential term $V(\phi)$; the Einstein equations reduce to the partial differential system

$$\frac{1}{2}\partial_t k^\alpha_\alpha + \frac{1}{4} k^\beta_\alpha k^\alpha_\beta = \chi \left[-\left(4{u_0}^2 - 1\right)\frac{\epsilon}{3} - \frac{1}{2}(\partial_t \phi)^2 + V(\phi) \right] \tag{63a}$$

$$\frac{1}{2}\left(k^\beta_{\alpha;\beta} - k^\beta_{\beta;\alpha}\right) = \chi \left(\frac{4}{3}\epsilon u_\alpha u_0 + \frac{1}{c}\partial_\alpha \phi \partial_t \phi \right) \tag{63b}$$

$$\frac{1}{2\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma} k^\beta_\alpha\right) + P^\beta_\alpha = \chi \left[\gamma^{\beta\sigma}\left(\frac{4}{3}\epsilon u_\alpha u_\sigma + \partial_\alpha \phi \partial_\sigma \phi \right) + \left(\frac{\epsilon}{3} + V(\phi) \right) \delta^\beta_\alpha \right] \tag{63c}$$

where, as usual, χ is the Einstein constant $\chi = \dfrac{8\pi G}{c^4}$ (we take $c = 1$), obvious notation for derivatives.

The partial differential equation describing the scalar field $\phi(t, x^\gamma)$ dynamics, deeply coupled to the Einstein ones, reads as

$$\partial_{tt}\phi + \frac{1}{2} k^\alpha_\alpha \partial_t \phi - \gamma^{\alpha\beta}\phi_{;\alpha\beta} + \frac{dV}{d\phi} = 0 \tag{64}$$

and finally the hydrodynamic equations accounting for the matter evolution are explicitly [27]

$$\frac{1}{\sqrt{\gamma}}\partial_t\left(\sqrt{\gamma}\epsilon^{3/4}u_0\right)+\frac{1}{\sqrt{\gamma}}\partial_\alpha(\sqrt{\gamma}\epsilon^{3/4}u^\alpha)=0 \tag{65a}$$

$$4\epsilon\left(\frac{1}{2}\partial_t {u_0}^2+u^\alpha\partial_\alpha u_0+\frac{1}{2}k_{\alpha\beta}u^\alpha u^\beta\right)=\left(1-{u_0}^2\right)\partial_t\epsilon-u_0u^\alpha\partial_\alpha\epsilon \tag{65b}$$

$$4\epsilon\left(u_0\partial_t u_\alpha+u^\beta\partial_\beta u_\alpha+\frac{1}{2}u^\beta u^\gamma\partial_\alpha\gamma_{\beta\gamma}\right)=-u_\alpha u_0\partial_t\epsilon+\left(\delta^\beta_\alpha-u_\alpha u^\beta\right)\partial_\beta\epsilon\,. \tag{65c}$$

Any kind of matter described by a perfect fluid energy-momentum tensor with equation of state $p=w\epsilon$ (w is a constant), $w\neq 0$, is dynamically equivalent to a scalar field $\psi(t,x^\gamma)$ with Lagrangian density

$$\mathcal{L}=\frac{1}{2}\sqrt{-g}\left(g^{ik}\partial_i\psi\partial_k\psi\right)^{\frac{1}{2}\left(\frac{1}{w}+1\right)} \tag{66}$$

once identified

$$\epsilon\equiv\frac{1}{2w}\left(g^{ik}\partial_i\psi\partial_k\psi\right)^{\frac{1}{2}\left(\frac{1}{w}+1\right)}\;, \tag{67a}$$

$$p\equiv\frac{1}{2}\left(g^{ik}\partial_i\psi\partial_k\psi\right)^{\frac{1}{2}\left(\frac{1}{w}+1\right)}\;, \tag{67b}$$

$$u_i\equiv\frac{\partial_i\psi}{\sqrt{g^{ik}\partial_i\psi\partial_k\psi}}\,, \tag{67c}$$

where g_{ik} $(i,k=0,1,2,3)$ is the four-dimensional covariant metric. The considered (Klein-Fock) scalar field ϕ $(w=1)$ corresponds to a perfect fluid with equation of state $p=\epsilon$, as well as the ultra-relativistic matter ($p=\frac{\epsilon}{3}$) is dynamically equivalent to a scalar field ψ, described by the above Lagrangian density in the case $w=\frac{1}{3}$.
The Einstein equations follow by the variational principle

$$\delta S=\delta\int\sqrt{-g}\left\{R-\chi\left[g^{ik}\partial_i\phi\partial_k\phi+\left(g^{ik}\partial_i\psi\partial_k\psi\right)^2\right]\right\}d^4x=0 \tag{68}$$

where R is the four-dimensional curvature scalar.
The quasi-isotropic solution near the cosmological singularity, as seen in Section **6** (and in [27]), refers to a Taylor expansion of the three-dimensional metric time dependence as (see 56)

$$\gamma_{\alpha\beta}(t,x^\gamma)=\sum_{n=0}^{\infty}{a^{(n)}}_{\alpha\beta}(x^\gamma)\left(\frac{t}{t_0}\right)^n \tag{69}$$

where

$$a^{(n)}_{\alpha\beta}(x^\gamma)\equiv\left.\frac{\partial^n\gamma_{\alpha\beta}}{\partial t^n}\right|_{t=0}{t_0}^n \tag{70}$$

in which t_0 is an arbitrarily fixed instant of time ($t \ll t_0$) and the existence of the singularity implies $a^{(0)}_{\alpha\beta} \equiv 0$. In [27] only the first two terms appear, *i.e.*

$$\gamma_{\alpha\beta} = a^{(1)}_{\alpha\beta}\frac{t}{t_0} + a^{(2)}_{\alpha\beta}\left(\frac{t}{t_0}\right)^2 . \tag{71}$$

The presence of the scalar field permits to relax the assumption of expandability in integer powers.
In order to introduce in a quasi-isotropic scenario (eventually inflationary, see below Section **9**) small inhomogeneous corrections to leading order, we require a three-dimensional metric tensor having the following structure

$$\begin{aligned}\gamma_{\alpha\beta}\left(t,x^{\gamma}\right) &= a^2(t)\xi_{\alpha\beta}\left(x^{\gamma}\right) + b^2(t)\theta_{\alpha\beta}\left(x^{\gamma}\right) + \mathcal{O}\left(b^2\right) = \\ &= a^2(t)\Big[\xi_{\alpha\beta}\left(x^{\gamma}\right) + \eta(t)\theta_{\alpha\beta}\left(x^{\gamma}\right) + \mathcal{O}\left(\eta^2\right)\Big] ,\end{aligned} \tag{72}$$

where we defined $\eta \equiv \dfrac{b^2}{a^2}$ and suppose η to satisfy the condition

$$\lim_{t\to\infty} \eta(t) = 0 . \tag{73}$$

In the limit of the considered approximation, the inverse three-metric reads

$$\gamma^{\alpha\beta}\left(t,x^{\gamma}\right) = \frac{1}{a^2(t)}\Big(\xi^{\alpha\beta}\left(x^{\gamma}\right) - \eta(t)\theta^{\alpha\beta}\left(x^{\gamma}\right) + \mathcal{O}\left(\eta^2\right)\Big) , \tag{74}$$

where $\xi^{\alpha\beta}$ denotes the inverse matrix of $\xi_{\alpha\beta}$ and assumes a metric role, *i.e.*we have

$$\xi^{\beta\gamma}\xi_{\alpha\gamma} = \delta^{\beta}_{\alpha} , \qquad \theta^{\alpha\beta} = \xi^{\alpha\gamma}\xi^{\beta\delta}\theta_{\gamma\delta} . \tag{75}$$

The covariant and contravariant three-metric expressions lead to the important explicit relations

$$k^{\beta}_{\alpha} = 2\frac{\dot a}{a}\delta^{\beta}_{\alpha} + \dot\eta\theta^{\beta}_{\alpha} \quad \Rightarrow \quad k^{\alpha}_{\alpha} = 6\frac{\dot a}{a} + \dot\eta\theta , \qquad \theta \equiv \theta^{\alpha}_{\alpha} . \tag{76}$$

Since the fundamental equality $\partial_t(\ln\gamma) = k^{\alpha}_{\alpha}$ holds, then we immediately get

$$\begin{aligned}\gamma = ja^6e^{\eta\theta} \quad \Rightarrow \quad \sqrt{\gamma} &= \sqrt{j}a^3e^{\frac{1}{2}\eta\theta} \sim \\ &\sim \sqrt{j}a^3\left(1 + \frac{1}{2}\eta\theta + \mathcal{O}(\eta^2)\right) ,\end{aligned} \tag{77}$$

once defined $j \equiv \det\xi_{\alpha\beta}$.

The Landau-Raychaudhury theorem [25] applied to the present case implies the condition

$$\lim_{t\to 0} a(t) = 0 . \tag{78}$$

The set of field equations (63) is solved retaining only the terms linear in η and its time derivatives and neglecting all terms containing spatial derivatives of the dynamical variables, in order to obtain asymptotic solutions in the limit $t \to 0$ and finally checking the self-consistence of the approximation scheme. The possibility to neglect the potential term $V(\phi)$ is not ensured by the field equations but is based on the idea that, in an inflationary scenario, the scalar field potential energy becomes dynamically relevant only during the "slow-rolling phase", far from the singularity, while the kinetic term asymptotically dominates. With this in mind the solution for $a(t)$ is found to be

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{3}}, \tag{79}$$

where t_0 is an integration constant, while for $\eta(t)$ it is

$$\eta(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}, \tag{80}$$

and for u_α

$$u_\alpha(t, x^\gamma) = v_\alpha(x^\gamma)\left(\frac{t}{t_0}\right)^{\frac{1}{3}} + \mathcal{O}\left(\frac{t}{t_0}\right), \tag{81}$$

respectively; hence we get for the tensor $\theta_{\alpha\beta}(x^\gamma)$ the expression

$$\theta_{\alpha\beta} = \frac{\rho}{3+4v^2}\Big[\left(1-2v^2\right)\xi_{\alpha\beta} + 10 v_\alpha v_\beta\Big] \quad \Rightarrow \quad \theta = \rho\,, \tag{82}$$

where $\rho(x^\gamma)$ denotes an arbitrary function of the spatial coordinates.
The energy density of the ultra-relativistic matter, to leading order, has the expression

$$\epsilon(t, x^\gamma) = \frac{5\rho(x^\gamma)}{3\chi\Big[3+4v^2(x^\gamma)\Big]{t_0}^{2/3}t^{4/3}} + \mathcal{O}\left(\frac{t}{t_0}\right), \tag{83}$$

and, once integrated the scalar field equation (64), they provide

$$\phi(t, x^\gamma) = \sqrt{\frac{2}{3\chi}}\left[\ln\left(\frac{t}{t_0}\right) - \frac{3}{4}\left(\frac{t}{t_0}\right)^{\frac{2}{3}}\rho(x^\gamma) + \sigma(x^\gamma)\right] + \mathcal{O}\left(\frac{t}{t_0}\right) \tag{84}$$

where $\sigma(x^\gamma)$ is an arbitrary function of the spatial coordinates.
Finally, equation (63c) yields the expression for the functions v_α in terms of ρ and of the spatial gradient $\partial_\alpha\sigma$ as

$$v_\alpha = -\frac{3(3+4v^2)}{10\rho\sqrt{1+v^2}} t_0 \partial_\alpha\sigma\,, \tag{85a}$$

$$v^2 = \frac{24\tau^2 - 1 + \sqrt{1-12\tau^2}}{2(1-16\tau^2)} \tag{85b}$$

where τ represents the quantity

$$\tau \equiv \frac{3t_0}{10\rho}\sqrt{\xi^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma}\,. \tag{85c}$$

The particular and simple case $\sigma \equiv 0$, in correspondence to which $v^2 \equiv 0$ ($v_\alpha \equiv 0$) leads to the solutions

$$\theta_{\alpha\beta} = \frac{1}{3}\rho(x^\gamma)\xi_{\alpha\beta} \tag{86a}$$

$$\epsilon(t, x^\gamma) = \frac{5}{9\chi}\rho(x^\gamma)\frac{1}{{t_0}^{2/3}t^{4/3}} + \mathcal{O}\left(\frac{t}{t_0}\right) \tag{86b}$$

$$\phi(t, x^\gamma) = \sqrt{\frac{2}{3\chi}}\ln\left(\frac{t}{t_0}\right) - \frac{3}{4}\sqrt{\frac{2}{3\chi}}\left(\frac{t}{t_0}\right)^{2/3}\rho(x^\gamma) + \mathcal{O}\left(\frac{t}{t_0}\right) \tag{86c}$$

$$u_\alpha(t, x^\gamma) = \frac{3}{8}\partial_\alpha\ln\big(\rho(x^\gamma)\big)t + \mathcal{O}\left(\frac{t}{t_0}\right)\,. \tag{86d}$$

Finally we obtain the three-dimensional metric tensor as

$$\gamma_{\alpha\beta}(t, x^\gamma) = \left(\frac{t}{t_0}\right)^{2/3}\left[1 + \left(\frac{t}{t_0}\right)^{2/3}\frac{\rho(x^\gamma)}{3}\right]\xi_{\alpha\beta}(x^\gamma) + \mathcal{O}\left(\frac{t}{t_0}\right)\,. \tag{86e}$$

On the basis of equations (86), the hydrodynamic ones (65) reduce to an identity in the considered approximation.

The solution here shown is completely self-consistent up to first-two orders in time and contains five physically arbitrary functions of the spatial coordinates: three out of the six functions $\xi_{\alpha\beta}$ (the remaining three of them can be fixed by pure spatial coordinates transformations), the spatial scalar $\rho(x^\gamma)$ and the function $\sigma(x^\gamma)$.

The independence among the functions $\xi_{\alpha\beta}$, ρ and σ implies the existence of a quasi-isotropic dynamics in correspondence to an arbitrary spatial distribution of ultra-relativistic matter.
The kinetic term of the scalar field behaves, to leading order, as $\sim a^{-6}$ and therefore, in the limit $a \to 0$, dominates over the ultra-relativistic energy density; in fact, the latter term diverges only as $\sim a^{-4}$ and the spatial curvature as $\sim a^{-2}$, making them negligible.

Furthermore, for a generic equation of state $p = w\epsilon$ the corresponding matter energy density behaves asymptotically as $\sim a^{-3(1+w)}$, but the above dynamical scheme is not qualitatively affected when considering values of w in the range $-\frac{1}{3} < w < 1$ instead of the ultra-relativistic case $w = \frac{1}{3}$.

8. – Quasi-isotropic Inflationary Solution

In this Section we find a solution for a quasi-isotropic inflationary Universe which allows to introduce in the problem a certain degree of inhomogeneity [14]. We consider a model which generalizes the (flat) FLRW one by introducing a first-order inhomogeneous term, whose dynamics is induced by an effective cosmological constant. The three-metric tensor consists of a dominant term, corresponding to an isotropic-like component, while the amplitude of the first-order one is controlled by a "small" function $\eta(t)$.

In a Universe filled with ultra-relativistic matter and a real self-interacting scalar field, we discuss the resulting dynamics, up to first order in η, when the scalar field performs a slow roll on a plateau of a symmetry breaking configuration and induces an effective cosmological constant.

We show how the spatial distribution of the ultra relativistic matter and of the scalar field admits an arbitrary form but nevertheless, due to the required inflationary e-folding, it cannot play a serious dynamical role in tracing the process of structures formation (via the Harrison–Zeldovich spectrum). As a consequence, we find reinforced the idea that the inflationary scenario is incompatible with a classical origin of the large scale structures.

8˙1. *Quasi-isotropic Inflation and Density Perturbation.* – As we have seen, the inflationary model [11, 7] is, up to now, the most natural and complete scenario to make account of the problems outstanding in the Standard Cosmological Model, like the horizons and flatness paradoxes [24] (for pioneer works on inflationary scenario and the spectrum of gravitational perturbation, see also [37, 38]); indeed such a dynamical scheme on one hand is able to justify the high isotropy of the cosmic microwaves background radiation (characterized by temperature fluctuations $\mathcal{O}(10^{-4})$ [9]) and, on the other one, provides a mechanism for generating a (scale invariant) spectrum of inhomogeneous perturbations (via the scalar field quantum fluctuations).
Moreover, as shown in [23, 39], a slow-rolling phase of the scalar field allows to connect the generic inhomogeneous Mixmaster dynamics [3, 13] with a later quasi-isotropic Universe evolution, in principle compatible with the actual cosmological picture, [41].
With respect to this, we investigate the dynamics performed by small inhomogeneous corrections to a leading order metric, during inflationary expansion.

The model presented in [14] has the relevant feature to contain inhomogeneous corrections to a flat FLRW Universe, which in principle could take a role to understand the process of structure formation, even in presence of an inflationary behaviour; however, a careful analysis of our result prevents this possibility in view of the strong inflationary e-folding, so confirming the expected incompatibility between an inflationary scenario and a classical origin of the Universe clumpyness.

In what follows, we will use the quasi isotropic solution which was introduced in [27] (see Section **6**) as the simplest, but rather general, extension of the FLRW model; for a discussion of the quasi isotropic solution in the framework of the "long-wavelength" approximation see [40], while for the implementation of such a solution after inflation [18] to a generic equation of state and to the case of two ideal hydrodynamic fluid see [19] and [20], respectively.
In the previous Section **7** this solution was discussed in the presence of a real scalar field kinetic energy, leading to a power-law solution for the three-metric, and predicted interesting features for the ultra-relativistic matter dynamics.

We analyse here the opposite dynamical scheme, *i.e.*when the scalar field undergoes a slow-rolling phase since the effective cosmological constant dominates its kinetic energy. We provide, up to first-two orders of approximation and in a synchronous reference, a detailed description of the three-metric, of the scalar field and of the ultra relativistic matter dynamics, showing that the volume of the Universe expands exponentially and induces a corresponding exponential decay (as the inverse fourth power of the cosmic scale factor), either of the three-metric corrections, as well as of the ultra relativistic matter (the same behaviour characterizes roughly even the scalar field inhomogeneities). It is remarkable that the spatial dependence of such component is described by a function which remains an arbitrary degree of freedom; in spite of such freedom in fixing the

primordial spectrum of inhomogeneities, due to the inflationary e-folding, we show there is no chance that, after the de Sitter phase, such relic perturbations can survive enough to trace the large scale structures formation by an Harrison–Zeldovich spectrum.
This behaviour suggests that the spectrum of inhomogeneous perturbations [29] cannot arise directly by the classical field nature, but by its quantum dynamics.

Finally, we recall that the presence of the kinetic term of a scalar field, here regarded as negligible, induces, near enough to the singularity, a deep modification of the general cosmological solution, leading to the appearance of a dynamical regime, during which the three spatial directions behave monotonically [4, 6] point by point in space.

8˙2. *Inhomogeneous Perturbations from an Inflationary Scenario.* – The theory of inflation is based on the idea that during the Universe evolution a phase transition takes place (for instance associated with a spontaneous symmetry breaking of a Grand Unification model of strong and electroweak interactions) which induces an effective cosmological constant dominating the expansion dynamics. As a result, an exponential expansion of the Universe arises and, under a suitable fine-tuning of the parameters, it is able to "stretch" so strongly the geometry that the *horizon* and *flatness paradoxes* of the SCM are naturally solved.
In the *new inflation* theory (see Section **4**˙1), the Universe undergoes a de Sitter phase when the scalar field performs a "slow-rolling" behaviour over a very flat region of the potential between the false and true vacuum. The exponential expansion ends with the scalar field falling down in the potential well associated to the real vacuum and the scalar field dies via damped (by the expansion of the Universe and particles creation) oscillations which reheat the cold Universe left by the de Sitter expansion (the relativistic particles temperature is proportional to the inverse scale factor). Indeed, the decay of this super-cooled bosons condensate into relativistic particles – as a typical irreversible process – generates a huge amount of entropy, which allows to account for the present high value ($\sim \mathcal{O}(10^{88})$) of the Universe entropy per comoving volume.

Apart from the transition across the potential barrier between false and true vacuum, which takes place in general via a tunnelling, the whole inflationary dynamics can be satisfactorily described via a classical *uniform* scalar field $\phi = \phi(t)$. The assumption that the field behaves in a classical way is supported by its bosonic and cosmological nature, but the existence of quantum fluctuations of the field within the different inflationary "bubbles" leads to relax the hypothesis of dealing with a perfectly uniform scalar field.

In general, when analysing density perturbations, it is convenient to introduce the dimensionless quantity [24, 33]

$$\delta\rho(t, x^\gamma) \equiv \frac{\Delta\rho(t, x^\gamma)}{\bar{\rho}} = \frac{\rho - \bar{\rho}}{\bar{\rho}}\,, \tag{87}$$

where $\bar{\rho}$ denotes the mean density and $\gamma = 1, 2, 3$. The best formulation of the density perturbations theory is obtained expanding $\delta\rho$ in its Fourier components, or modes,

$$\delta\rho_k = \frac{1}{(2\pi)^3}\int d^3x\, e^{ik_\alpha x^\alpha}\,\delta\rho(t, x^\gamma)\,. \tag{88}$$

As long as the perturbations are in the linear regime, *i.e.*$\delta\rho_k \ll 1$, it is possible to follow appropriately the dynamics of each mode with wave number k, which corresponds to a wavelength $\lambda = \dfrac{2\pi}{k}$; however, the *physical* size of the perturbations in an expanding

Universe evolves via the *cosmic scale factor* which from now on we write as $a(t)$, in order to avoid ambiguities.

Since in the Standard Cosmological Model, the "Hubble radius" scales as $H^{-1} \propto t$, while $a(t) \propto t^n$ with $n < 1$, then every perturbation, now inside the Hubble radius, was outside it at some earlier time. We stress how the perturbations with a physical size smaller or greater than the Hubble radius have a very different dynamics, respectively, the former ones being affected by the action of the microphysics processes.

In the case of an inflationary scenario, the situation is quite different. Since during the de Sitter phase the Hubble radius remains constant while the cosmic scale factor "blows up" exponentially, hence all cosmologically interesting scales have crossed the horizon twice, *i.e.*the perturbations begin sub-horizon sized, cross the Hubble radius during inflation and later cross back again inside the horizon.
This feature has a strong implication on the initial spectrum of density perturbations predicted by inflation. We present a qualitative argument to understand how this spectrum can be generated.

During inflation, the density perturbations are expected to arise from the quantum mechanical fluctuations of the scalar field ϕ; these are, as usual, decomposed in their Fourier components $\delta\phi_k$, *i.e.*

$$\delta\phi_k = \frac{1}{(2\pi)^3} \int d^3x \, e^{ik_\alpha x^\alpha} \delta\phi(t, x^\gamma)\,. \tag{89}$$

The spectrum of quantum mechanical fluctuations of the scalar field is defined as

$$(\Delta\phi)^2_k \equiv \frac{1}{\mathcal{V}} \frac{k^3}{2\pi^2} \left|\delta\phi_k\right|^2 \,, \tag{90}$$

where $\mathcal{V}$ denotes the comoving volume. For a massless minimally coupled scalar field in a de Sitter space-time, which approximates very well the real physical situation during the Universe exponential expansion, it is well known that (see [24])

$$(\Delta\phi)^2_k = \left(\frac{H}{2\pi}\right)^2 \,, \tag{91}$$

then the mean square fluctuation of ϕ takes the form

$$(\Delta\phi)^2 = \frac{1}{(2\pi)^3 V} \int d^3k \, |\delta\phi_k|^2 = \int \left(\frac{H}{2\pi}\right)^2 d(\ln k)\,. \tag{92}$$

Since H is a constant during the de Sitter phase, each mode k contributes roughly the same amplitude to the mean square fluctuation. Indeed, the only dependence on k takes place in the logarithmic term, but the modes of cosmological interest lay between 1 Mpc and 3000 Mpc (it is commonly adopted the convention to set the actual cosmic scale factor equal to unity), corresponding to a logarithmic interval of less than an order of magnitude.
Thus we can conclude that any mode k crosses the horizon having almost a constant amplitude $\delta\phi_k \simeq H/2\pi$. A delicate question concerns the mechanism by which such

quantum fluctuations of the scalar field achieve a classical nature [35]; here we simply observe how each mode k, once reached a classical stage, is governed by the dynamics

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = 0\,; \tag{93}$$

according to this equation, super-horizon modes $k \ll aH$ ($i.e.\lambda_{phys} \gg H^{-1}$) admit the trivial dynamics (93) with $\delta\phi_k \sim$ const.. This simple analysis implies the important feature that any mode re-enters the horizon with roughly the same amplitude it had at the first horizon crossing. The spectrum of perturbations so generated is then induced into the relativistic energy density coming from the reheating phase, associated with the bosons decay; since then on, the evolution of the perturbation spectrum follows a standard paradigm.

The density perturbations discussed so far are related to the scalar field by

$$\delta\rho = \frac{\partial V}{\partial\phi}\delta\phi\,, \tag{94}$$

where, because of the potential is very flat during inflation, $\partial V/\partial\phi$ is approximatively constant and then

$$\delta\rho \simeq \text{const.} \times \delta\phi\,. \tag{95}$$

The spectrum of density perturbations has a Harrison-Zeldovich form, characterized by a constant amplitude: this is a very generic prediction of inflation, based on the features of the potential flatness common to nearly all inflationary models. On the other hand, the spectrum amplitude is model dependent and accurate measures could discriminate between the various models.

Finally, we discuss the Gaussian distribution of the quantum mechanical fluctuations generated by the inflation: as long as the field ϕ is minimally coupled, it has a low self interaction and each mode fluctuates independently; hence, since the fluctuations we actually observe are the sum of many of the quantum ones, their distribution can be expected to be Gaussian (as it should be for the sum of many independent variables). This is reflected on the distribution of temperature fluctuations of the CMBR as a powerful test inflation. In a detailed analysis by [42], 82 (even if not independent) hypothesis tests for Gaussianity are implemented, showing how the MAXIMA map is consistent with Gaussianity on angular scales between 10' and 5°, where deviations are most likely to occur.

8˙3. *Geometry, Matter and Scalar Field Equations.* – When matter is represented as a perfect fluid with ultra relativistic equation of state $p = \frac{\epsilon}{3}$ and in presence of a scalar field $\phi(t, x^\gamma)$ with a potential term $V(\phi)$ the Einstein equations read as

$$R_i^k = \chi \sum_{(z)=m,\phi} \left(T_i^{k\,(z)} - \frac{1}{2}\delta_i^k T_l^{l\,(z)} \right) \tag{96}$$

where $T_i^{k\,(m)}$ and $T_i^{k\,(\phi)}$ indicate, respectively, the energy-momentum tensor of the matter and the scalar field, $i.e.$explicitly as in (63); this set is coupled to the dynamics of the

scalar field $\phi(t, x^\gamma)$ given by (64) and to the hydrodynamic equations (65), which take into account for the matter evolution. In view of the chosen feature for (96), equation (65a) doesn't contain any spatial gradients of the three-metric tensor and of the scalar field. This scheme is completed by observing how it can be made covariant with respect to coordinate transformations of the form

$$t' = t + f(x^\gamma)\,, \qquad x^{\alpha\prime} = x^{\alpha\prime}(x^\gamma) \tag{97}$$

f being a generic space dependent function.

9. – Quasi Homogeneous Inflationary Solution

In order to introduce in a quasi isotropic (inflationary) scenario small inhomogeneous corrections to the leading order, we require a three-dimensional metric tensor having the structure as in (72)-(75) and (76)-(77).

The field equations (63) are analysed via the standard procedure of constructing asymptotic solutions in the limit $t \to \infty$, retaining only terms linear in η and its time derivatives and by verifying *a posteriori* the self-consistency of the approximation scheme, *i.e.*the neglected terms to be really of higher order in time [14].

In the quasi-isotropic approach, we assume that the scalar field dynamics in the plateau region is governed by a potential

$$V(\phi) = \Lambda + K(\phi)\,, \qquad \Lambda = \text{const.} \tag{98}$$

where Λ is the dominant term and $K(\phi)$ is a small correction to it. The role of K, as shown in the following, is to contain inhomogeneous corrections via the ϕ-dependence; the functional form of K can be any one of the most common inflationary potentials, as they appear near the flat region for the evolution of ϕ.
What follows remains valid, for example, for the relevant cases of the quartic and Coleman–Weinberg expressions already introduced in some detail in Section **4**˙1

$$K(\phi) = \begin{cases} -\dfrac{\lambda}{4}\phi^4\,, & \lambda = \text{const.} \\ B\phi^4\left[\ln\left(\dfrac{\phi^2}{\sigma^2}\right) - \dfrac{1}{2}\right]\,, & \sigma = \text{const.}\,, \end{cases} \tag{99}$$

viewed as corrections to the constant Λ term, although explicit calculations are developed below only for the first case.

Our inflationary solution is obtained under the standard requirements

$$\frac{1}{2}\left(\partial_t\phi\right)^2 \ll V\left(\phi\right) \tag{100a}$$

$$\mid \partial_{tt}\phi \mid \ll \mid k^\alpha_\alpha \partial_t\phi \mid\ . \tag{100b}$$

The approximations (100) and the substitution of (76) reduce the scalar field equation (64) to

$$\left(3\frac{\dot a}{a} + \frac{1}{2}\dot\eta\theta\right)\partial_t\phi - \lambda\phi^3 = 0 \tag{101}$$

where the contribution of the ϕ spatial gradient is assumed to be negligible.

Similarly, the quasi-isotropic approach (in which the inhomogeneities become relevant only for the next-to-leading order), neglecting the spatial derivatives in (65a), leads to

$$\sqrt{\gamma}\epsilon^{3/4}u_0 = l(x^\gamma) \quad \Rightarrow \quad \epsilon \sim \frac{l^{4/3}}{j^{2/3}a^4{u_0}^{4/3}}\left(1 - \frac{2}{3}\eta\theta + \mathcal{O}(\eta^2)\right)\ ,, \tag{102}$$

where $l(x^\gamma)$ denotes an arbitrary function of the spatial coordinates.

Let us now face in the same approximation scheme the analysis of the Einstein equations (63). Taking into account (100a), equation (63a) to first order in η reads

$$3\frac{\ddot{a}}{a} + \left[\frac{1}{2}\ddot{\eta} + \frac{\dot{a}}{a}\dot{\eta}\right]\theta - \chi\Lambda = -\chi\frac{\epsilon}{3}\left(3 + 4u^2\right) \tag{103}$$

having set

$$u^2 \equiv \frac{1}{a^2}\xi^{\alpha\beta}u_\alpha u_\beta \quad \Rightarrow \quad u_0 = \sqrt{1 + u^2}\,. \tag{104}$$

Equation (63c) reduces to the form

$$\begin{aligned}\frac{2}{3}\left(a^3\right)_{,tt}\,\delta^\beta_\alpha + \left(a^3\ \eta_{,t}\right)_{,t}\,\theta^\beta_\alpha + \frac{1}{3}\Big[\left(a^3\right)_{,t}\,\eta\Big]_{,t}\,\theta\delta^\beta_\alpha + aA^\beta_\alpha = \\ = \chi\Big[\frac{1}{a^2}\Big(\xi^{\beta\gamma} - \eta\theta^{\beta\gamma}\Big)\frac{4}{3}\epsilon u_\alpha u_\gamma + \Big(\frac{\epsilon}{3} + \Lambda\Big)\,\delta^\beta_\alpha\Big]2a^3\left(1 + \frac{\eta\theta}{2}\right)\,,\end{aligned} \tag{105}$$

where we adopted the notation $(\ )_{,t} \equiv d/dt$ and $(\ )_{,tt} \equiv d^2/dt^2$ for simplicity of writing. In this expression, the spatial curvature term reads, to leading order, as

$$P^\beta_\alpha(t, x^\gamma) = \frac{1}{a^2(t)}A^\beta_\alpha(x^\gamma)\,, \tag{106}$$

where $A_{\alpha\beta}(x^\gamma) = \xi_{\beta\gamma}A^\gamma_\alpha$ denotes the Ricci tensor corresponding to $\xi_{\alpha\beta}(x^\gamma)$.

The trace of (105) provides the additional relation

$$2\left(a^3\right)_{,tt} + \left(a^3\eta\right)_{,tt}\theta + aA^\alpha_\alpha = \chi\left[\frac{\epsilon}{3}\left(3 + 4u^2\right) + 3\Lambda\right]2a^3\left(1 + \frac{\eta\theta}{2}\right)\,. \tag{107}$$

Comparing (103) with the trace (107), via their common term $(3+4u^2)\epsilon/3$, and estimating the different orders of magnitude, we get the following equations:

$$\left(a^3\right)_{,tt} + 3a^2a_{,tt} - 4\chi a^3\Lambda = 0 \tag{108a}$$

$$A_{\alpha\beta} = 0 \tag{108b}$$

$$3\left(a^3\eta\right)_{,tt} + 3a^3\eta_{,tt} + 2\left(a^3\right)_{,t}\eta_{,t} + 9a^2\eta\ a_{,tt} - 12\chi a^3\Lambda\eta = 0\,. \tag{108c}$$

Since (108b) implies the vanishing of the three-dimensional Ricci tensor and this condition corresponds to the vanishing of the Riemann tensor too, then we can conclude that the obtained Universe is flat to leading order, *i.e.*

$$\xi_{\alpha\beta} = \delta_{\alpha\beta} \qquad \Rightarrow \qquad j = 1. \tag{109}$$

Equation (108a) admits the expanding solution

$$a(t) = a_0 \exp\left(\frac{\sqrt{3\chi\Lambda}}{3} t\right) \tag{110}$$

a_0 being the initial value of the scale factor amplitude, taken at the instant $t = 0$ when the de Sitter phase starts.

Expression (110) for $a(t)$ substituted in (108c) yields the differential equation for η

$$\ddot{\eta} + \frac{4}{3}\sqrt{3\chi\Lambda}\ \dot{\eta} = 0\,, \tag{111}$$

whose only solution satisfying the limit (73) reads as

$$\eta(t) = \eta_0 \exp\left(-\frac{4}{3}\sqrt{3\chi\Lambda}\ t\right) \qquad \Rightarrow \qquad \eta = \eta_0 \left(\frac{a_0}{a}\right)^4\,, \tag{112}$$

and, of course, we require $\eta_0 \ll a_0$.

Equations (102) and (103), in view of the solutions (110) for $a(t)$ and (112) for $\eta(t)$, are matched for consistency by posing

$$\begin{aligned} u_\alpha(t, x^\gamma) &= v_\alpha(x^\gamma) + \mathcal{O}\left(\eta^2\right) \\ (u_0)^2 &= 1 + \mathcal{O}\left(\frac{1}{a^2}\right) \approx 1\,, \end{aligned} \tag{113}$$

and

$$\epsilon = -\frac{4}{3}\Lambda\eta\theta\,, \tag{114}$$

respectively, which implies $\theta < 0$ for each point of the allowed domain of the spatial coordinates. The comparison between (102) and (114) leads to the explicit expression for $l(x^\gamma)$ in terms of θ

$$l(x^\gamma) = \left(\frac{4}{3}\Lambda\eta_0 {a_0}^4\right)^{3/4} (-\theta)^{3/4}\ . \tag{115}$$

Defining the auxiliary tensor with unit trace $\Theta_{\alpha\beta}(x^\gamma) \equiv \theta_{\alpha\beta}/\theta$, the above analysis permits to obtain for it from (105) the expression

$$\Theta_\alpha^\beta = \frac{\delta_\alpha^\beta}{3}\,. \tag{116}$$

By (101), the explicit form for a expanded in η yields the first-two leading orders of approximation for the scalar field

$$\phi\left(t,x^{\gamma}\right)=\mathcal{C}\sqrt{\frac{t_r}{t_r-t}}\left(1-\frac{1}{4\sqrt{3\chi\Lambda}}\frac{\eta}{t_r-t}\theta\right)\,, \qquad t_r=\frac{\sqrt{3\chi\Lambda}}{\mathcal{C}^2 2\lambda}\,, \tag{117}$$

where $\mathcal{C}$ is an integration constant; finally, equation (63c) provides v_α in terms of θ

$$v_\alpha=-\frac{3}{4}\frac{1}{\sqrt{3\chi\Lambda}}\partial_\alpha \ln\mid\theta\mid\ . \tag{118}$$

On the basis of (116)-(118), the hydrodynamic equations (65) reduce to an identity to leading order; in fact such equations contain the energy density of the ultra relativistic matter, known only to first order (the higher one of the Einstein equations). Therefore it makes no sense to take into account higher order contributions, coming from those equations.

As soon as (t_r-t) is sufficiently large, the solution here constructed can be easily checked to be completely self-consistent to all the calculated orders of approximation in time and contains one physically arbitrary function of the spatial coordinates, $\theta(x^\gamma)$ which indeed, being a three-scalar, is not affected by spatial coordinate transformations. In particular, the terms quadratic in the spatial gradients of the scalar field are of order

$$\left(\partial_\alpha\phi\right)^2\approx\mathcal{O}\left(\frac{\eta^2}{a^2}\frac{1}{\left(t_r-t\right)^3}\right) \tag{119}$$

and therefore can be neglected with respect to all the inhomogeneous ones.
Such a solution fails when t approaches t_r and ts validity requires the de Sitter phase to end (with the fall of the scalar field in the true potential vacuum) when t is still much smaller than t_r.

9˙1. *Physical considerations*. – The peculiar feature of the solution constructed lies in the independence of the function θ which, from a cosmological point of view, implies the existence of a quasi-isotropic inflationary solution in correspondence with an arbitrary spatial distribution of ultra-relativistic matter and of the scalar field.

We get an inflationary picture from which the Universe outcomes with the appropriate standard features, but in presence of a suitable spectrum of *classical* perturbations as due to the small inhomogeneities which can be modelled according to an Harrison–Zeldovich spectrum; in fact, expanding the function θ in Fourier series as

$$\theta(x^\gamma)=\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\tilde{\theta}\left(\vec{k}\right)e^{i\vec{k}\cdot\vec{x}}d^3k\,, \tag{120}$$

we can impose an Harrison–Zeldovich spectrum by requiring

$$\mid\tilde{\theta}\mid^2=\frac{Z}{\mid k\mid^{3/2}}\,,\qquad Z=\text{const.}. \tag{121}$$

However, the following three points have to be taken into account to give a complete picture for our analysis:

(i) limiting (as usual) our attention to leading order, the validity of the slow-rolling regime is ensured by the natural conditions

$$\mathcal{O}\left(\sqrt{\chi\Lambda}(t-t_r)\right) \ll 1\,, \qquad \lambda \gg \mathcal{O}(\chi^2\Lambda)\,, \tag{122}$$

which respectively translate (100b) and (100a);

(ii) denoting by t_i and t_f respectively the beginning and the end of the de Sitter phase, we should have $t_r \gg t_f$ and the validity of our solution is guaranteed if

(a) the flatness of the potential is preserved, *i.e.*$\lambda\phi^4 \ll \Lambda$: such a requirement coincides, as it should, with the second of inequalities (122);

(b) given Δ as the width of the flat region of the potential, we require that the de Sitter phase ends before t becomes comparable with t_r, *i.e.*

$$\phi(t_f) - \phi(t_i) \sim \sqrt{\frac{\sqrt{\chi\Lambda}}{\lambda}\frac{t_f - t_i}{t_r^{3/2}}} \sim \mathcal{O}(\Delta)\,, \tag{123}$$

where we expanded the solution to first order in $t_{i,f}/t_r$; via the usual position $(t_f - t_i) \sim \mathcal{O}(10^2)/\sqrt{\chi\Lambda}$, the relation (123) becomes a constraint for the integration constant t_r.

(iii) In order to get an inflationary scenario, able to overcome the shortcomings present in the SCM, we need an exponential expansion sufficiently strong. For instance we have to require a region of space corresponding to a cosmological horizon $\mathcal{O}(10^{-24}cm)$ when the de Sitter phase starts to cover now all the actual Hubble horizon $\mathcal{O}(10^{26}cm)$; the redshift at the end of the de Sitter phase is $z \sim \mathcal{O}(10^{24})$, then we should require $a_f/a_i \sim e^{60} \sim \mathcal{O}(10^{26})$.
Let us estimate the density perturbations (inhomogeneities) at the (matter-radiation) decoupling age ($z \sim \mathcal{O}(10^4)$) as $\delta_{in} \sim \mathcal{O}(10^{-4})$; if we start by this same value at the beginning of inflation (δ^i_{in}), we arrive at the end with $\delta^f_{in} \sim (\eta_f/\eta_i)\delta^i_{in} \sim \mathcal{O}(10^{-100})$. Though these inhomogeneities increase as z^2 once they are at scale greater than the horizon, nevertheless they reach only $\mathcal{O}(10^{-60})$ at the decoupling age.
This result provides support to the idea that the spectrum of inhomogeneous perturbations cannot have a classical origin in presence of an inflationary scenario.

In the considerations above developed, we regard the ratio of the inhomogeneous terms ϵ_f and ϵ_i as the quantity $\delta\rho$ and this assumption is (roughly) correct: after the reheating, the Universe is dominated by a homogeneous (apart from the quantum fluctuations) relativistic energy density ρ_r to which is superimposed the relic ϵ_f after inflation; therefore we have

$$\delta\rho = \frac{\epsilon_f}{\rho_r} = \frac{\epsilon_f}{\epsilon_i}\frac{\epsilon_i}{\rho_r} = \left(\frac{a_i}{a_f}\right)^4 \frac{\epsilon_i}{\rho_r}\,. \tag{124}$$

Hence our statement follows as soon as we observe that the inhomogeneous relativistic energy density before the inflation ϵ_i and the uniform one ρ_r, generated by the reheating process, differ by only some orders of magnitude.

REFERENCES

[1] A. Albrecht and P. J. Steinhardt, *Phys. Rev. Lett.*, **48**, (1982) 1220.

[2] R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation: an introduction to current research* (1962), eds. I. Witten and J. Wiley, New York.

[3] V.A. Belinski, I.M. Khalatnikov and E.M. Lifshitz (1970) *Adv. Phys.*, **19**, 525.

[4] V.A. Belinski and I.M. Khalatnikov, *Sov. Phys. JETP*, (1973) **36**, 591.

[5] V.A. Belinski, I.M. Khalatnikov and E.M. Lifshitz (1982) *Adv. Phys.*, **31**, 639.

[6] B.K. Berger, *Phys.Rev.D*, (2000) **61**, 023508 (available gr-qc/9907083).

[7] S. Coleman and E. Weinberg, *Phys. Rev. D*, (1973) **7**, 1888.

[8] C.B. Collins and S.W. Hawking, (1973) *Astrophys. Journ.* **180**, 317

[9] P. de Bernardis et al., *Astrophys.J.* (2002) **564**, 559, (available astro-ph/0105296).

[10] L.P. Grischchuk, A.G. Doroshkevich and V.M. Yudin (1975) *Sov. Phys. JETP*, **42**, n.6, 943.

[11] A.H. Guth, *Phys. Rev. D*, (1981) **23**, 347.

[12] G.P. Imponente, G. Montani (2001) *Phys. Rev. D*, **63**, 103501.

[13] G.P. Imponente and G. Montani *Int. Journ. Mod. Phys. D*, (2002) **11**, n.8, 1321, (available gr-qc/0106028).

[14] G.P. Imponente and G. Montani *Int. Journ. Mod. Phys. D*, in press (available gr-qc/0307048) (2003).

[15] G.P. Imponente and G. Montani (2003) submitted.

[16] G.P. Imponente and G. Montani in “Proceedings of the Tenth Marcel Grossmann Meeting on General Relativity 2003” eds. V.G. Gurzadyan, R.T.Jantzen and R.Ruffini, (World Scientific, Singapore) in preparation

[17] G.P. Imponente and G. Montani in “Proceedings of the Tenth Marcel Grossmann Meeting on General Relativity 2003” eds. V.G. Gurzadyan, R.T.Jantzen and R.Ruffini, (World Scientific, Singapore) in preparation

[18] I.M. Khalatnikov et al., *Pis'ma Zh. Eksp. Teor. Fiz.* **38**, 79 (1983) [JETP Lett., **38**, 91 (1983)]; *J.Stat.Phys.* **38**, 97 (1985).

[19] I.M. Khalatnikov, A.Yu. Kamenshchik and A.A. Starobinsky *Class. and Quantum Grav.*,(2002) **19** 3845 (gr-qc/0204045).

[20] I.M. Khalatnikov, A.Yu. Kamenshchik, M. Martellini and A.A. Starobinsky, *Journ. Cosm. and Astropart. Phys.*, (2003) **0303**, 001 (gr-qc/0301119).

[21] A.A. Kirillov, *Zh. ksp. Teor. Fiz.* **103**, 721 (1993) [Sov. Phys. JETP 76, 355 (1993)];

[22] A.A. Kirillov and A.A. Kochnev, *JETP Lett.*, **46**, (1987), 435.

[23] A.A. Kirillov and G. Montani, *Phys. Rev. D*, (2002) **66**, 064010.

[24] E.W. Kolb and M.S. Turner, *The Early Universe*, (1990) (Adison-Wesley, Reading).

[25] L.D. Landau and E.M. Lifshitz, *“Classical Theory of Fields”*, (Mir), (1975), 4th Edition, Addison-Wesley, New York.

[26] A.R. Liddle, *Phys. Lett.*, **220B**, (1989) 502.

[27] E.M. Lifshitz and I.M. Khalatnikov, *Adv. Phys.*, (1963) **12**, 185.

[28] A.D. Linde, *Phys. Lett.*, **108B**, (1982) 389.

[29] C.P. Ma and E. Bertschinger, *Astrophys. J.*, (1995) **455**, 7.

[30] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, ed.by W H Freeman and C., New York (1973)

[31] G. Montani, *Class. and Quantum Grav.*, (1999) **16**, 723.

[32] G. Montani, *Class. and Quantum Grav.*, (2000) **17**, 2205.

[33] T. Padmanabham, *Structure Formation in the Universe*, (1993), (Cambridge University Press).
[34] R. Penrose (1982), *Proc.R.Soc.Lond.* **A 381**, 53.
[35] D. Polarski, A.A. Starobinsky, *Class. Quantum Grav.*, (1996) **13**, 377.
[36] D.N. Spergel, *Astrophys.J. Suppl. Series* (2003) **148**, 175
[37] A.A. Starobinsky, *Phys. Lett. B*, (1980) **91**, 99.
[38] A.A. Starobinsky, *JETP Lett.*, (1979) **30**, 682.
[39] A.A. Starobinsky, *Pis'ma Zh. Eksp. Teor. Fiz.* (1983) **37**, 55.
[40] K. Tomita and N. Deruelle, *Phys. Rev. D*, (1994) **50**, n.12, 7216.
[41] H. van Elst, P. Dunsby and R. Tavakol, *Gen. Rel. Grav.* **27**, 171-191 (1995).
[42] J.H.P. Wu et al., *Tests for Gaussianity of the MAXIMA-1 CMB Map*, (2001) available on astro-ph/0104428.